Transmission Problems Arising in Czochralski Process of Crystal Growth

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TRANSMISSION PROBLEMS ARISING IN CZOCHRALSKI PROCESS OF CRYSTAL GROWTH

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Abstract. We consider a transmission problem of two parabolic PDEs as a mathematical model for the so-called Czochralski crystal growth process. Originally the Czochralski process is a three-phase system of solid-liquid-gas and involves the three interfaces of solid-liquid, liquid-gas and solid-gas which are to be determined as a part of the solution from the mathematical point of view. Nevertheless, in this paper, assuming that the solid-gas and liquid-gas interfaces are known, we propose a transmission problem for the couple of a linear heat equation in the gas region and a nonlinear degenerate parabolic equation in the material region as a simplified mathematical model of the Czochralski process. Moreover we prove the existence and uniqueness of a weak solution in the variational sense.

1. Introduction

The Czochralski pulling method is wildly used for the production of a column of single crystal from the melt substance. The idea for pulling method due to Czochralski is quite simple. A crucible equipped with a heating system contains the melt substance and a pul-rod with seed crystal, which moves vertically and rotates flexibly, is positioned above the crucible (see Fig.1). The rod is dipped into the melt, and then lifted slowly with an appropriate speed v_p so that a meniscus surface is formed below the seed crystal and the melt attached to the crystal solidifies continuously. By controlling some thermal situations in the process one obtains the growth of a single crystal column with a desired radius as well as a desired growth pattern of the solid-liquid interface and temperature pattern in the crystal in order to improve the crystal quality.

In such a model of crystal growth the shape of crystal is determined by three kinetic equations of three interfaces between solid-liquid, liquid-gas and solid-gas. But, in this paper we suppose that the crystal radius is controlled to be constant and the trijunction curve on which three interfaces meet is prescribed, too. This might be designed by a good choice of the pulling velocity. As a consequence we assume that the movement of the material domain is prescribed.

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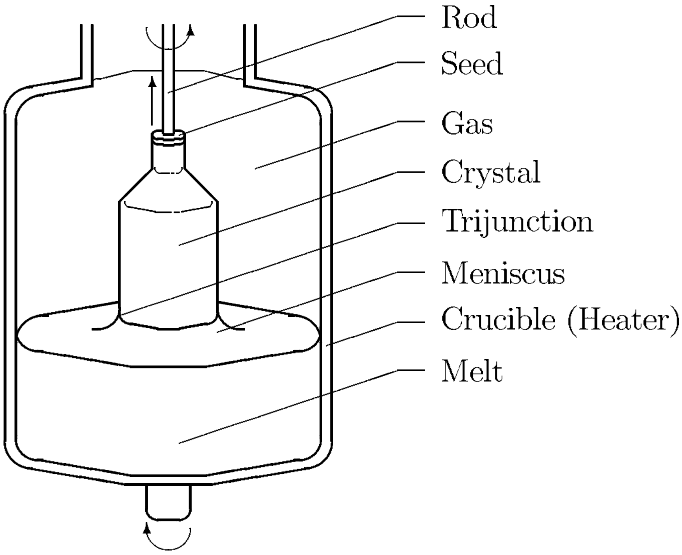

Fig.1

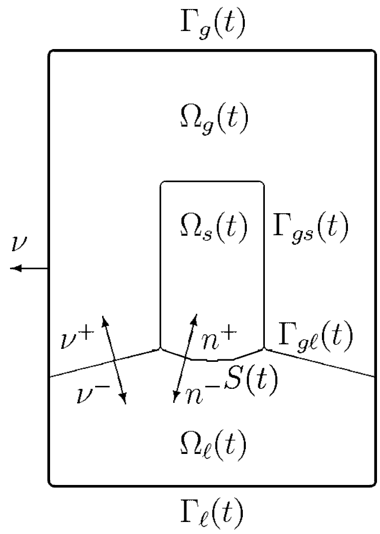

Fig.2

We use the following notation (see Fig.2): For $0 < T < \infty$ and $t \in [0, T]$,

$\Omega_\ell(t)$: liquid (melt) region,
$\Omega_s(t)$: solid (crystal) region,
$\Omega_g(t)$: gas region,
$S(t)$: solid-liquid interface,
$\Omega := \{\Omega_\ell(t) \cup \Omega_s(t) \cup S(t)\} \cup \Omega_g(t) \cup \{\Gamma_{g\ell}(t) \cup \Gamma_{gs}(t)\}$,
$\Gamma(= \partial\Omega) := \Gamma_g(t) \cup \Gamma_\ell(t)$,
$\Omega_m(t) := \Omega_\ell(t) \cup \Omega_s(t) \cup S(t)$: material region,
$\Gamma_m(t)(= \partial\Omega_m(t)) := \Gamma_\ell(t) \cup \{\Gamma_{g\ell} \cup \Gamma_{gs}(t)\}$,
$\Gamma_{gm}(t) := \Gamma_{g\ell}(t) \cup \Gamma_{gs}(t)$,
$\nu = \nu(t, x)$: 3-dimensional unit vector normal to Γ at $x \in \Gamma$,
$\nu^+ = \nu^+(t, x)$: 3-dimensional unit vector normal to $\Gamma_{gm}(t)$, pointing to $\Omega_g(t)$,
$\nu^- = \nu^-(t, x)$: 3-dimensional unit vector normal to $\Gamma_{gm}(t)$, pointing to $\Omega_m(t)$,
$n^+ = n^+(t, x)$: 3-dimensional unit vector normal to $S(t)$, pointing to $\Omega_s(t)$,
$n^- = n^-(t, x)$: 3-dimensional unit vector normal to $S(t)$, pointing to $\Omega_\ell(t)$,
$Q := (0, T) \times \Omega$,
$\Sigma := (0, T) \times \Gamma$,
$Q_i := \bigcup_{t \in (0,T)} \{t\} \times \Omega_i(t) \quad i = m, \ell, s, g$,
$\Sigma_i := \bigcup_{t \in (0,T)} \{t\} \times \Gamma_i(t) \quad i = m, \ell, g, gm, g\ell, gs$,
$S := \bigcup_{t \in (0,T)} \{t\} \times S(t)$.

Next, we denote by $v_{\Sigma_m} := v_{\Sigma_m}(t, x)$ the normal speed of $\Gamma_m(t)$ at $(t, x) \in \Sigma_m$. With this v_{Σ_m} the 4-dimensional unit vector outward normal to Σ_m at each $(t, x) \in \Sigma_m$ is given by

$$\vec{\nu}^+ := (\vec{\nu}_t^+, \vec{\nu}_x^+) = \frac{1}{(|v_{\Sigma_m}|^2 + 1)^{\frac{1}{2}}}(-v_{\Sigma_m}, \nu^+).$$

Similarly, with the normal speed $v_S := v_S(t, x)$ of $S(t)$ at $(t, x) \in S$, the 4-dimensional unit vector normal to S, pointing to the solid region, is given by

$$\vec{n}^+ := (\vec{n}_t^+, \vec{n}_x^+) = \frac{1}{(|v_S|^2+1)^{\frac{1}{2}}}(-v_S, n^+).$$

It is easily understood that by the crystal growth the shape of material domain $\Omega_m(t)$ changes with time and it yields a 3-dimensional convective vector field $\mathbf{v} := \mathbf{v}(t,x)$ in Q. The determination of $\mathbf{v}$ is also one of the important questions in the mathematical modeling of the Czochralski process of crystal growth. It seems that $\mathbf{v}$ obeys mainly the pulling velocity v_p in the crystal and is a solution of the incompressible Navier-Stokes (or simply Stokes) equation in the melt (see [3, 5]). Nevertheless, in this paper, we assume that the convective field $\mathbf{v}$ is prescribed, too, at least satisfying that

$$\mathrm{div}\mathbf{v} = 0 \quad \text{in } Q, \tag{1.1}$$

$$\mathbf{v}\cdot\nu^+ = v_{\Sigma_m} \quad \text{on } \Sigma_{gm}(t), \tag{1.2}$$

$$\mathbf{v}\cdot\nu = 0 \quad \text{on } \Sigma_i(t),\ i=\ell,g. \tag{1.3}$$

Now, from the usual energy balance laws the following system is derived, where $\theta_i := \theta_i(t,x)$ is the temperature on Q_i, $i=\ell,s,g$.

$$c_i(\theta_{i,t} + \mathbf{v}\cdot\nabla\theta_i) - k_i\Delta\theta_i = f \qquad \text{in } Q_i,\ i=\ell,s,g, \tag{1.4}$$

$$\theta_\ell = \theta_s = 0, \quad k_\ell\frac{\partial\theta_\ell}{\partial n^+} + k_s\frac{\partial\theta_s}{\partial n^-} = L(\mathbf{v}\cdot n^+ - v_S) \quad \text{on } S, \tag{1.5}$$

$$k_i\theta_i = k_g\theta_g, \quad k_i\frac{\partial\theta_i}{\partial\nu^+} + k_g\frac{\partial\theta_g}{\partial\nu^-} = 0 \quad \text{on } \Sigma_{gi},\quad i=\ell,s, \tag{1.6}$$

subject to the boundary conditions

$$k_i\frac{\partial\theta_i}{\partial\nu} + n_0k_i\theta_i = p \quad \text{on } \Sigma_i,\quad i=\ell,g, \tag{1.7}$$

and initial conditions

$$\theta_i(0,\cdot) = \theta_{i0} \quad \text{on } \Omega_i(0),\ \ i=\ell,s,g, \qquad S(0)=S_0. \tag{1.8}$$

Here note that (1.5) is called the Stefan conditions, where the phase change temperature is supposed to be 0 for simplicity, and (1.6) is called the transmission conditions between the material and gas; $c_i,\ k_i.\ i=\ell,s,g$, and L are positive constants which are the specific heats, the heat conductivities and latent heat, respectively; f is a given heat source on Q, p is the boundary data prescribed on Σ; $\theta_{i0}, i=\ell,s,g$, are the initial temperature on $\Omega_i(0)$ and S_0 is the initial location of the solid-liquid interface, satisfying that

$$\theta_0 > 0 \quad \text{on } \Omega_\ell(0), \quad \theta_0 < 0 \quad \text{on } \Omega_s(0), \quad \theta_0 = 0 \quad \text{on } S_0.$$

As is well known, by using the enthalpy we reformulate the above system as a weak variational form. In this paper we prove its well-posedness by using the abstract theory of time-dependent subdifferential operators (cf. [1, 4, 7, 8, 9, 16]).

2. Weak formulation

The enthalpy u is defined as follows:

$$u := \begin{cases} c_s\theta & \text{if } x \in \Omega_m(t),\ \theta < 0, \\ [0, L] & \text{if } x \in \Omega_m(t),\ \theta = 0, \\ c_\ell\theta + L & \text{if } x \in \Omega_m(t),\ \theta > 0, \\ c_g\theta & \text{if } x \in \overline{\Omega_g(t)}. \end{cases}$$

Moreover we define a function $\beta(t, x, r)$ on $Q \times \mathbf{R}$ by

$$\beta(t,x,r) := \begin{cases} \beta_1(r) \ := \left\{ \begin{array}{ll} \dfrac{k_s}{c_s} r & \text{if } \ r < 0, \\ 0 & \text{if } \ 0 \le r \le L, \\ \dfrac{k_\ell}{c_\ell}(r - L) & \text{if } \ r > L, \end{array} \right\} & \text{if } x \in \Omega_m(t), \\ \beta_2(r) \ := \ \dfrac{k_g}{c_g} r & \text{if } x \in \overline{\Omega_g(t)}. \end{cases}$$

Then $\beta(t, x, r)$ is a non-decreasing Lipschitz continuous function in r on $\mathbf{R}$ for a.e. $(t, x) \in Q$.

By using the enthalpy u our problem (1.4)-(1.8) is reformulated as an initial-boundary value problem for a degenerate parabolic equation in a cylindrical domain Q of the following form

$$(\mathrm{E}) \quad \begin{cases} u_t - \Delta\beta(u) + \mathbf{v} \cdot \nabla u = f \quad \text{in } Q, & (2.1) \\ \dfrac{\partial\beta(u)}{\partial\nu} + n_0\beta(u) = p \quad \text{on } \Sigma, \\ u(0) = u_0 \quad \text{on } \Omega, \end{cases}$$

where $\beta(u) := \beta(t, x, u)$ and

$$u_0 := c_s\theta_0\chi_{\Omega_s(0)} + (c_\ell\theta_0 + L)\chi_{\Omega_\ell(0)} + c_g\theta_0\chi_{\Omega_g(0)}.$$

with characteristic functions $\chi_{\Omega_i(0)}$ of $\Omega_i(0), i = \ell, s, g$. In fact, multiply equations (1.4) by any test function $\eta \in C^2(\overline{Q})$ with $\eta(T) := \eta(T, \cdot) = 0$ on Ω, and then integrate the resultants over Q_i, $i = \ell, s, g$, respectively, to get the following identities

$$\int_{Q_i} c_i\theta_t\eta dxdt - \int_{Q_i} k_i\Delta\theta\eta dxdt + \int_{Q_i} c_i(\mathbf{v} \cdot \nabla\theta)\eta dxdt = \int_{Q_i} f\eta dxdt, \quad i = \ell, s, g, \quad (2.2)$$

where the subindex i of θ_i is omitted, since no confusion arises. With the help of (1.2), (1.3) and the Green-Stokes' formula as well as the relations $d\Sigma_i = (|v_{\Sigma_m}|^2+1)^{\frac{1}{2}}d\Gamma_i(t)dt, i = gm, g\ell, gs$, $d\Sigma_i = d\Gamma_i(t)dt, i = \ell, g$, and $dS = (|v_S|^2 + 1)^{\frac{1}{2}}dS(t)dt$, we have

$$\int_{Q_\ell} c_\ell\theta_t\eta dxdt + \int_{Q_s} c_s\theta_t\eta dxdt + \int_{Q_g} c_g\theta_t\eta dxdt$$

$$
\begin{aligned}
= \; & \int_{\Sigma_{g\ell}} c_\ell \theta \eta \vec{\nu}_t^{+} d\Sigma_{g\ell} + \int_{\Omega_\ell(0)} c_\ell \theta_0 \eta(0)(-1) dx - \int_{Q_\ell} c_\ell \theta \eta_t dx dt \\
& + \int_{\Sigma_{gs}} c_s \theta \eta \vec{\nu}_t^{+} d\Sigma_{gs} + \int_{\Omega_s(0)} c_s \theta_0 \eta(0)(-1) dx - \int_{Q_s} c_s \theta \eta_t dx dt \\
& + \int_{\Sigma_{g\ell}} c_g \theta \eta \vec{\nu}_t^{-} d\Sigma_{g\ell} + \int_{\Sigma_{gs}} c_g \theta \eta \vec{\nu}_t^{-} d\Sigma_{gs} + \int_{\Omega_g(0)} c_g \theta_0 \eta(0)(-1) dx - \int_{Q_g} c_g \theta \eta_t dx dt \\
= \; & -\int_Q u \eta_t dx dt - \int_\Omega u_0 \eta(0) dx \\
& + \int_{Q_\ell} L \eta_t dx dt + \int_{\Omega_\ell(0)} L \eta(0) dx + \int_{\Sigma_{g\ell}} c_\ell \theta \eta (-v_{\Sigma_m}) d\Gamma_{g\ell}(t) dt \\
& + \int_{\Sigma_{gs}} c_s \theta \eta (-v_{\Sigma_m}) d\Gamma_{gs}(t) dt + \int_{\Sigma_{g\ell}} c_g \theta \eta (v_{\Sigma_m}) d\Gamma_{g\ell}(t) dt + \int_{\Sigma_{gs}} c_g \theta \eta (v_{\Sigma_m}) d\Gamma_{gs}(t) dt,
\end{aligned}
$$

and by (1.6) we have

$$
\begin{aligned}
& -\int_{Q_\ell} k_\ell \Delta \theta \eta dx dt - \int_{Q_s} k_s \Delta \theta \eta dx dt - \int_{Q_g} k_g \Delta \theta \eta dx dt \\
= \; & -\int_0^T \left\{ \int_{\Gamma_\ell(t)} k_\ell \frac{\partial \theta}{\partial \nu} \eta d\Gamma_\ell(t) + \int_{\Gamma_{g\ell}(t)} k_\ell \frac{\partial \theta}{\partial \nu^+} \eta d\Gamma_{g\ell}(t) + \int_{S(t)} k_\ell \frac{\partial \theta}{\partial n^+} \eta dS(t) \right\} dt \\
& + \int_{Q_\ell} k_\ell (\nabla \theta \cdot \nabla \eta) dx dt \\
& - \int_0^T \left\{ \int_{\Gamma_{gs}(t)} k_s \frac{\partial \theta}{\partial \nu^+} \eta d\Gamma_{gs}(t) + \int_{S(t)} k_s \frac{\partial \theta}{\partial n^-} \eta dS(t) \right\} dt + \int_{Q_s} k_s (\nabla \theta \cdot \nabla \eta) dx dt \\
& - \int_0^T \left\{ \int_{\Gamma_g(t)} k_g \frac{\partial \theta}{\partial \nu} \eta d\Gamma_g(t) + \int_{\Gamma_{g\ell}(t)} k_g \frac{\partial \theta}{\partial \nu^-} \eta d\Gamma_{g\ell}(t) + \int_{\Gamma_{gs}(t)} k_g \frac{\partial \theta}{\partial \nu^-} \eta d\Gamma_{gs}(t) \right\} dt \\
& + \int_{Q_g} k_g (\nabla \theta \cdot \nabla \eta) dx dt \\
= \; & \int_Q \nabla \beta(u) \cdot \nabla \eta dx dt - \int_\Sigma \frac{\partial \beta(u)}{\partial \nu} \eta d\Gamma dt - \int_0^T \int_{S(t)} \left(k_\ell \frac{\partial \theta}{\partial n^+} + k_s \frac{\partial \theta}{\partial n^-} \right) \eta dS(t) dt.
\end{aligned}
$$

Moreover it follows from (1.1) that

$$
\begin{aligned}
& \int_{Q_\ell} c_\ell (\mathbf{v} \cdot \nabla \theta) \eta dx dt + \int_{Q_s} c_s (\mathbf{v} \cdot \nabla \theta) \eta dx dt + \int_{Q_g} c_g (\mathbf{v} \cdot \nabla \theta) \eta dx dt \\
= \; & \int_0^T \left\{ \int_{\Gamma_\ell(t)} c_\ell \theta (\mathbf{v} \cdot \nu) \eta d\Gamma_\ell(t) + \int_{\Gamma_{g\ell}(t)} c_\ell \theta (\mathbf{v} \cdot \nu^+) \eta d\Gamma_{g\ell}(t) \right\} dt - \int_{Q_\ell} c_\ell \theta (\mathbf{v} \cdot \nabla \eta) dx dt \\
& + \int_0^T \left\{ \int_{\Gamma_{gs}(t)} c_s \theta (\mathbf{v} \cdot \nu^+) \eta d\Gamma_{gs}(t) \right\} dt - \int_{Q_s} \theta (\mathbf{v} \cdot \nabla \eta) dx dt \\
& + \int_0^T \left\{ \int_{\Gamma_g(t)} c_g \theta (\mathbf{v} \cdot \nu) \eta d\Gamma_g(t) + \int_{\Gamma_{g\ell}(t)} c_g \theta (\mathbf{v} \cdot \nu^-) \eta d\Gamma_{g\ell}(t) \right. \\
& \left. + \int_{\Gamma_{gs}(t)} c_g \theta (\mathbf{v} \cdot \nu^-) \eta d\Gamma_{gs}(t) \right\} dt - \int_{Q_g} c_g \theta (\mathbf{v} \cdot \nabla \eta) dx dt \\
= \; & -\int_Q u (\mathbf{v} \cdot \nabla \eta) dx dt + \int_{Q_\ell} L (\mathbf{v} \cdot \nabla \eta) dx dt
\end{aligned}
$$

$$+\int_{\Sigma_{g\ell}} c_\ell\theta(\mathbf{v}\cdot\nu^+)\eta d\Gamma_{g\ell}(t)dt+\int_{\Sigma_{gs}} c_s\theta(\mathbf{v}\cdot\nu^+)\eta d\Gamma_{gs}(t)dt$$
$$+\int_{\Sigma_{g\ell}} c_g\theta(\mathbf{v}\cdot\nu^-)\eta d\Gamma_{g\ell}(t)dt+\int_{\Sigma_{gs}} c_g\theta(\mathbf{v}\cdot\nu^-)\eta d\Gamma_{gs}(t)dt.$$

Using (1.2) and (1.3) we get

$$-\int_Q u\eta_t dxdt+\int_Q \nabla\beta(u)\cdot\nabla\eta dxdt-\int_\Sigma \frac{\partial\beta(u)}{\partial\nu}\eta d\Gamma dt-\int_Q u(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$+\int_{Q_\ell} L\eta_t dxdt+\int_{\Omega_\ell(0)} L\eta(0)dx-\int_S\left(k_\ell\frac{\partial\theta}{\partial n^+}+k_s\frac{\partial\theta}{\partial n^-}\right)\eta dS(t)dt+\int_{Q_\ell} L(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$=\int_Q f\eta dxdt+\int_\Omega u_0\eta(0)dx.$$

Moreover on account of the Stefan condition (1.5), we get the following equation:

$$\begin{aligned}
&\int_{Q_\ell} L\eta_t dxdt+\int_{\Omega_\ell(0)} L\eta(0)dx-\int_S\left(k_\ell\frac{\partial\theta}{\partial n^+}+k_s\frac{\partial\theta}{\partial n^-}\right)\eta dS(t)dt+\int_{Q_\ell} L(\mathbf{v}\cdot\nabla\eta)dxdt\\
=\ &\int_{\Sigma_{g\ell}} L\eta\vec{\nu}_t^+ d\Sigma_{g\ell}+\int_S L\eta\vec{n}_t^+ dS+\int_{\Omega_\ell(0)} L\eta(0)dx\\
&-\int_S L(\mathbf{v}\cdot n^+-v_S)dS(t)dt+\int_{\Sigma_{g\ell}} L\eta(\mathbf{v}\cdot\nu^+)d\Gamma_{g\ell}(t)dt+\int_S L\eta(\mathbf{v}\cdot n^+)dS(t)dt\\
=\ &0.
\end{aligned}$$

Therefore, using (1.7), we arrive at the following equality

$$-\int_Q u\eta_t dxdt+\int_Q \nabla\beta(u)\cdot\nabla\eta dxdt+n_0\int_\Sigma \beta(u)\eta d\Gamma dt-\int_Q u(\mathbf{v}\cdot\nabla\eta)dxdt$$
$$=\int_Q f\eta dxdt+\int_\Sigma p\eta d\Gamma dt+\int_\Omega u_0\eta(0)dx\quad\text{for all }\eta\in W,\tag{2.3}$$

where $W:=\{\eta\in H^1(Q);\ \eta(T)=0\text{ on }\Omega\}$. This is the variational form of (E) which is regarded as a weak formulation of (1.4)-(1.8) as usual. Problem (E) is called the enthalpy formulation of (1.4)-(1.8).

Remark 2.1 In the transmission condition (1.6), equalities $k_i\theta_i=k_g\theta_g$, $i=\ell,s$, imply that the temperature is possibly discontinuous on Σ_{gm}, since $k_i\neq k_g$ in general. Therefore, this may not be acceptable from some physical points of view. When the continuity of the temperature is required in Q, $k_i\theta_i=k_g\theta_g$ should be replaced by $\theta_i=\theta_g$ on Σ_{gm}. In this case, the weak variational identity is different from the above (2.3) and we need a different mathematical treatment of the problem and this will be studied in a forthcoming paper of the authors. Also, some mathematical modelings of this sort of crystal growth have been discussed by Pawlow [13] in more general settings.

For the solvability of two-phase Stefan problems without convection in cylindrical domains, the time-dependent subdifferential operator theory was skillfully applied by Damlamian [4]. The case of non-cylindrical domains with convection was treated by Fukao, Kenmochi and Pawlow [6] by applying the theory of quasi-linear equations of parabolic type.

3. Statements of main results

In this section we formulate our main result with some functional settings. First we introduce a function space $V := H^1(\Omega)$ equipped with norm

$$|u|_V^2 := |\nabla u|_{L^2(\Omega)}^2 + n_0 \int_\Gamma |u|^2 d\Gamma \quad \text{for all } u \in V,$$

which is an equivalent norm to the standard one of $H^1(\Omega)$. The dual space of V is denoted by V^* and the duality pairing between V^* and V is denoted by $\langle \cdot, \cdot \rangle_{V^*,V}$; note that if $u^* \in L^2(\Omega)$, then

$$\langle u^*, u \rangle_{V^*,V} = (u^*, u)_{L^2(\Omega)}, \quad \text{for all } u \in V,$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the inner product in $L^2(\Omega)$, The duality mapping $F : V \to V^*$ is defined by

$$\langle F u_1, u_2 \rangle_{V^*,V} := \int_\Omega \nabla u_1 \cdot \nabla u_2 dx + n_0 \int_\Gamma u_1 u_2 d\Gamma \quad \text{for all } u_2 \in V.$$

Moreover, defining an inner product $(\cdot, \cdot)_{V^*}$ in V^* by

$$(u_1^*, u_2^*)_{V^*} := \langle u_1^*, F^{-1} u_2^* \rangle_{V^*,V}, \quad \text{for all } u_1^*, u_2^* \in V^*.$$

we see that V^* is a Hilbert space with the following dense and compact imbeddings:

$$V \hookrightarrow L^2(\Omega) \hookrightarrow V^*.$$

Now we give the definition of a weak solution of (E).

Definition 3.1 $u : [0, T] \to L^2(\Omega)$ is called a weak solution of (E) if u satisfies following conditions:

(D1) $u \in W^{1,2}(0, T; V^*) \cap L^\infty(0, T; L^2(\Omega))$.
(D2) $\beta(u) \in L^2(0, T; V)$.
(D3) u satisfies the variational identity (2.3).

Our main result is stated under the following geometric assumption $(\star)$ on the time dependence of the prescribed material domain $\Omega_m(t)$:

$(\star)$ Assume that Ω is a bounded domain in $\mathbf{R}^3$ with smooth boundary Γ and assume that there exists a one-parameter family of C^2-diffeomorphims $\Theta := \Theta(t, \cdot)$ on $\overline{\Omega}$, $0 \le t \le T$, such that $\Theta \in C^2(\overline{Q})$ and

$$\Theta(t, \overline{\Omega_i(t)}) = \overline{\Omega_i(0)}, \quad i = m, g, \ 0 \le t \le T.$$

Theorem 3.1 *Assume that condition* $(\star)$ *holds,* $f \in L^2(Q)$, $p \in C^1(\overline{\Sigma})$, $u_0 \in L^2(\Omega)$ *and* $\beta(u_0) \in H^1(\Omega)$. *Also assume that* $\mathbf{v} \in C^1(\overline{Q})^3$ *and* (1.1), (1.2) *and* (1.3) *hold. Then there exists one and only one weak solution* u *of* (E).

The proofs of existence and uniqueness of a weak solution of (E) will be given in sections 4 and 5, respectively.

4. Existence proof

Let us consider the existence question of a weak solution of (E). Our approach is based on the time-dependent subdifferential operator theory. To do so we define a function $\phi^t : L^2(\Omega) \to (-\infty, \infty]$, for each $t \in [0,T]$ by this formula:

$$\phi^t(z) = \begin{cases} \int_{\Omega_m(t)} \hat{\beta}_1(z(x))dx + \int_{\Omega_g(t)} \hat{\beta}_2(z(x))dx & \text{if} \quad z \in L^2(\Omega), \\ +\infty & \text{if} \quad z \in V^* \setminus L^2(\Omega), \end{cases}$$

where $\hat{\beta}_i(r) := \int_0^r \beta_i(s)ds, i = 1,2$. Clearly ϕ^t is a proper, lower semi-continuous and convex function on V^*. We denote by $\partial\phi^t$ the subdifferential of ϕ^t; by definition (cf. [2]), $z^* \in \partial\phi^t(z)$ in V^* means that

$$(z^*, w - z)_{V^*} \le \phi^t(w) - \phi^t(z), \quad \text{for all } w \in V^*.$$

The characterization of $\partial\phi^t$ is given in the following lemma.

Lemma 4.1 *Let $z \in D(\partial\phi^t)$ and $\beta(z) := \beta_1(z)\chi_{\Omega_m(t)} + \beta_2(z)\chi_{\Omega_g(t)}$. Then $z^* \in \partial\phi^t(z)$ in V^* if and only if*

$$(F^{-1}z^*, w)_{L^2(\Omega)} = (\beta(z), w)_{L^2(\Omega)} \quad \textit{for all } w \in L^2(\Omega). \tag{4.1}$$

Moreover, this is equivalent to the statement that $\beta(z) \in V$ and

$$\langle z^*, \eta \rangle_{V^*,V} = \int_\Omega \nabla\beta(z) \cdot \nabla\eta dx + n_0 \int_\Gamma \beta(z)\eta d\Gamma \quad \textit{for all } \eta \in V. \tag{4.2}$$

Proof. We have $z \in L^2(\Omega)$ by $D(\partial\phi^t) \subset D(\phi^t) = L^2(\Omega)$ and by definition

$$(z^*, w - z)_{V^*} \le \int_{\Omega_m(t)} \left(\hat{\beta}_1(w(x)) - \hat{\beta}_1(z(x))\right) dx + \int_{\Omega_g(t)} \left(\hat{\beta}_2(w(x)) - \hat{\beta}_2(z(x))\right) dx,$$
$$\text{for all } w \in L^2(\Omega).$$

Here, choose $z + \varepsilon(\tilde{w} - z), \varepsilon > 0$, as w in the above inequality for each $\tilde{w} \in L^2(\Omega)$, divide the both sides by ε and finally pass to the limit $\varepsilon \to 0$. Then we get

$$(z^*, \tilde{w} - z)_{V^*} \le (\beta_1(z)\chi_{\Omega_m(t)} + \beta_2(z)\chi_{\Omega_g(t)}, \tilde{w} - z)_{L^2(\Omega)} \quad \text{for all } \tilde{w} \in L^2(\Omega).$$

From the arbitrariness of $\tilde{w} \in L^2(\Omega)$ and the relation $(z^*, \tilde{w} - z)_{V^*} = (F^{-1}z^*, \tilde{w} - z)_{L^2(\Omega)}$ it follows that

$$(F^{-1}z^*, w)_{L^2(\Omega)} = (\beta_1(z)\chi_{\Omega_m(t)} + \beta_2(z)\chi_{\Omega_g(t)}, w)_{L^2(\Omega)} \quad \text{for all } w \in L^2(\Omega),$$

namely (4.1) holds. It is easy to show the converse assertion. Also, by the definition of the duality mapping F, (4.1) and (4.2) are equivalent to each other. □

Now we are ready to apply the existence result in [16] (or [7]) for nonlinear evolution equations including time-dependent subdifferential operators and small perturbations. Consider the following initial value problem (K):

$$\text{(K)} \quad \begin{cases} u_t(t) + \partial\phi^t(u(t)) + g(t, u(t)) \ni f^*(t) & \text{in } V^* \text{ for a.e. } t \in [0,T], \\ u(0) = u_0 \quad \text{in } V^*, \end{cases}$$

where $g(t,z) : L^2(\Omega) \to V^*$ is an operator defined by:

$$\langle g(t,z), \eta\rangle_{V^*,V} := -\int_\Omega z(x)(\mathbf{v}(t)\cdot\nabla\eta)(x)dx \quad \text{for all } \eta \in V.$$

for each $t \in [0,T]$ and $z \in L^2(\Omega)$. In the next lemma we check the time-dependence of ϕ^t and the conditions required in order to apply the abstract result in [16] (or [7]) to problem (K).

Lemma 4.2 (i) *There exists a positive constant* α_0 *with the following property: for each* $s, t \in [0,T], s \le t$ *and* $z \in D(\phi^s)$ *there exists* $z_1 \in D(\phi^t)$ *such that*

$$|z_1 - z|_{V^*} \le \alpha_0|t-s|(1+|\phi^s(z)|^{\frac{1}{2}}), \tag{4.3}$$

$$|\phi^t(z_1) - \phi^s(z)| \le \alpha_0|t-s|(1+|\phi^s(z)|). \tag{4.4}$$

Furthermore the set $L_r := \bigcup_{0\le t\le T}\{z \in V^*; \phi^t(z) + |z|^2_{V^*} \le r\}$ *is bounded in* $L^2(\Omega)$ *and hence relatively compact in* V^* *for each* $r \ge 0$.
(ii) *The operator* $g(t,\cdot) : D(g(t,\cdot)) = L^2(\Omega) \to V^*$, $t \in [0,T]$, *satisfies that*

$$|g(t,z)|^2_{V^*} \le C_0\phi^t(z) + C_1 \quad \textit{for all } t \in [0,T] \textit{ and all } z \in L^2(\Omega), \tag{4.5}$$

where C_0 *and* C_1 *are positive constants, and that if* $\{z_n\} \subset L^2(\Omega), \{t_n\} \subset [0,T], \{\phi^{t_n}(z_n)\}$ *is bounded,* $z_n \to z$ *in* V^* *and* $t_n \to t$, *as* $n \to \infty$, *then* $g(t_n, z_n) \to g(t,z)$ *weakly in* V^* *as* $n \to \infty$.
Proof. (i) By our geometric assumption $(\star)$ the mapping $\Theta_{t,s} := \Theta(s,\cdot)^{-1} \circ \Theta(t,\cdot)$ is a C^2-diffeomorphism on $\overline{\Omega}$ for any $s,t \in [0,T]$ such that $\Theta_{t,s}(\overline{\Omega_i(t)}) = \overline{\Omega_i(s)}$, $i = g, m$. Also there exist positive constants α_1 and α_2 such that

$$|\det.J_{\Theta_{t,s}}(x) - 1| \le \alpha_1|t-s|, \quad \text{for all } s,t \in [0,T], \text{and } x \in \Omega,$$

for each $\eta \in V$

$$|\eta\circ\Theta_{t,s} - \eta|_{L^2(\Omega)} \le \alpha_2|t-s||\eta|_V, \quad \text{for all } s,t \in [0,T],$$

(cf. Kenmochi [8; Lemma 3.2.2]), where $J_{\Theta_{t,s}}$ is the Jacobian of $\Theta_{t,s}$ and $\det.J_{\Theta_{t,s}}$ is its determinant.

Now given $z \in D(\phi^s) = L^2(\Omega)$ we take as z_1 the function $z\circ\Theta_{t,s} \in L^2(\Omega) = D(\phi^t)$. Then for each $\eta \in V$ we have:

$$\begin{aligned}
& \langle z_1 - z, \eta\rangle_{V^*,V} = (z_1 - z, \eta)_{L^2(\Omega)} \\
= & \int_\Omega z_1(x)\eta(x)dx - \int_\Omega z(x)\eta(x)dx \\
= & \int_\Omega z(\Theta_{t,s}(x))\eta(x)dx - \int_\Omega z(x)\eta(x)dx \\
= & \int_\Omega z(x)\eta(\Theta^{-1}_{t,s}(x))\det.J_{\Theta_{t,s}}(x)dx - \int_\Omega z(x)\eta(x)dx \\
\le & \int_\Omega |z(x)||\eta(\Theta_{s,t}(x))||\det.J_{\Theta_{s,t}}(x) - 1|dx + \int_\Omega |z(x)||\eta(\Theta_{s,t}(x)) - \eta(x)|dx \\
\le & \alpha_1|t-s|\int_\Omega |z(x)||\eta(\Theta_{s,t}(x))|dx + |z|_{L^2(\Omega)}|\eta\circ\Theta_{s,t} - \eta|_{L^2(\Omega)} \\
\le & \alpha_1|t-s||z|_{L^2(\Omega)}|\eta\circ\Theta_{s,t}|_{L^2(\Omega)} + \alpha_2|t-s||z|_{L^2(\Omega)}|\eta|_V.
\end{aligned}$$

Since $|\mathrm{det}.J_{\Theta_{s,t}}| \le 1 + \alpha_1 T$, we have $|\eta \circ \Theta_{s,t}|_{L^2(\Omega)} \le \alpha_3 |\eta|_{L^2(\Omega)}$, where $\alpha_3 := (1+\alpha_1 T)^{\frac{1}{2}}$. Therefore it follows from the above inequalities that

$$\langle z_1 - z, \eta \rangle_{V^*,V} \le \alpha_4 |t-s| |z|_{L^2(\Omega)} |\eta|_V, \tag{4.6}$$

where $\alpha_4 := \alpha_1\alpha_3 + \alpha_2$. By the definition of $\hat{\beta}_i$ there are positive constant c_β and c'_β such that

$$\hat{\beta}_i(r) \ge c_\beta |r|^2 - c'_\beta \quad \text{for all } r \in \mathbf{R} \text{ and } i = 1,2,$$

so that it is easy to find a positive constant c''_β such that

$$|w|_{L^2(\Omega)} \le c''_\beta (1 + |\phi^s(w)|^{\frac{1}{2}}) \quad \text{for all } w \in L^2(\Omega) \text{ and } s \in [0,T]. \tag{4.7}$$

Combining this with (4.6) we immediately get (4.3) with any positive constant α_0 with $\alpha_0 \ge \alpha_4 c''_\beta$.

Next, we recall

$$\begin{aligned} &\phi^t(z_1) - \phi^s(z) \\ &= \int_{\Omega_m(t)} \hat{\beta}_1(z_1(x))dx + \int_{\Omega_g(t)} \hat{\beta}_2(z_1(x))dx - \int_{\Omega_m(s)} \hat{\beta}_1(z(x))dx - \int_{\Omega_g(s)} \hat{\beta}_2(z(x))dx. \end{aligned}$$

Here

$$\begin{aligned} \int_{\Omega_i(t)} \hat{\beta}_j(z_1(x))dx - \int_{\Omega_i(s)} \hat{\beta}_j(z(x))dx &\le \int_{\Omega_i(t)} \hat{\beta}_j(z(\Theta_{t,s}(x))dx - \int_{\Omega_i(s)} \hat{\beta}_j(z(x))dx \\ &\le \int_{\Omega_i(s)} \left\{ \hat{\beta}_j(z(x))\mathrm{det}.J_{\Theta_{t,s}}(x) - \hat{\beta}_j(z(x)) \right\} dx \\ &\le \int_{\Omega_i(s)} \hat{\beta}_j(z(x)) |\mathrm{det}.J_{\Theta_{t,s}}(x) - 1| dx \\ &\le \alpha_1 |t-s| \int_{\Omega_i(s)} \hat{\beta}_j(z(x))dx, \end{aligned}$$

for $(i,j) = (m,1)$ and $(g,2)$. Now it is easy to see that (4.4) holds by taking α_0 so that $\alpha_0 > \alpha_1$. Also, for each $r \ge 0$ the boundedness of L_r in $L^2(\Omega)$ is immediately obtained from (4.7).

All assertions in (ii) are easily seen from the definition of $g(t,\cdot)$. □

Now, we define $f^* \in L^2(0,T;V^*)$ by

$$\langle f^*(t), \eta \rangle_{V^*,V} := \int_\Omega f(t,x)\eta(x)dx + \int_\Gamma p(t,x)\eta(x)d\Gamma \quad \text{for all } \eta \in V,$$

and consider the evolution equation (K). Applying an existence result of [16] (or see [7]), we see by Lemma 4.2 that there exists at least one solution u of (K) in $W^{1,2}(0,T;V^*) \cap L^\infty(0,T;L^2(\Omega))$. Moreover we see that this solution u is a weak solution of (E), too. In fact, since

$$f^*(t) - u_t(t) - g(t,u(t)) \in \partial\phi^t(u(t)) \quad \text{in } V^* \text{ for a.e. } t \in [0,T],$$

it follows from Lemma 4.1 that

$$F^{-1}(f^* - u_t - g(t,u)) = \beta(u) \in L^2(0,T;V),$$

and for each $\eta \in H^1(Q)$ with $\eta(T) = 0$

$$\begin{aligned}&\langle f^*(t) - u_t(t) - g(t, u(t)), \eta(t)\rangle_{V^*,V} \\ =\ & \int_\Omega \nabla\beta(u(t)) \cdot \nabla\eta(t)dx + n_0 \int_\Gamma \beta(u(t))\eta(t)d\Gamma.\end{aligned}$$

Therefore, integrating this in time on $[0, T]$ and using integration by parts, we see that u satisfies the variational identity (2.3). Thus u is a weak solution of (E) in the sense of Definition 3.1 and the existence proof of Theorem 3.1 has been accomplished.

5. Uniqueness proof

In this section we prove the uniqueness of a weak solution of (E) obtained in the previous section. The idea of our uniqueness proof is due to Ladyženskaja, Solonnikov and Ural'ceva [11; Chapter 3, Section 3], and this was also extensively used in Niezgódka and Pawlow [12], Rodrigues and Yi [15] and Rodrigues [14] for the uniqueness proof of generalized Stefan problems and continuous casting problems.

Let u_1 and u_2 be two weak solutions. Then, by taking the difference of their weak variational forms and using the Green's formula we have

$$\begin{aligned}&-\int_Q (u_1 - u_2)\eta_t dxdt - \int_Q (\beta(u_1) - \beta(u_2))\Delta\eta dxdt - \int_Q (u_1 - u_2)(\mathbf{v} \cdot \nabla\eta)dxdt \\ &+ \int_\Sigma (\beta(u_1) - \beta(u_2))\frac{\partial\eta}{\partial\nu}d\Gamma dt + n_0 \int_\Sigma (\beta(u_1) - \beta(u_2))\eta d\Gamma dt = 0 \qquad (5.1)\\ &\text{for all } \eta \in W.\end{aligned}$$

As usual, consider the function

$$b(t, x) := \begin{cases} \dfrac{\beta(u_1(t, x)) - \beta(u_2(t, x))}{u_1(t, x) - u_2(t, x)} & \text{if} \quad u_1(t, x) \neq u_2(t, x), \\ 0 & \text{if} \quad u_1(t, x) = u_2(t, x), \end{cases}$$

which is non-negative and bounded on Q. By virtue of (5.1)

$$-\int_Q (u_1 - u_2)\{\eta_t + b\Delta\eta + \mathbf{v} \cdot \nabla\eta\}dxdt + \int_\Sigma (\beta(u_1) - \beta(u_2))\left\{\frac{\partial\eta}{\partial\nu} + n_0\eta\right\}d\Gamma dt = 0 \qquad (5.2)$$

$$\text{for all } \eta \in W.$$

We now take smooth and strictly positive approximations b_ε of b such that

$$b \leq b_\varepsilon \quad \text{a.e. on } Q, \ \varepsilon \leq b_\varepsilon \leq B_0 \quad \text{a.e. on } Q, \qquad (5.3)$$

$$b_\varepsilon \to b \quad \text{a.e. on } Q \text{ as } \varepsilon \to 0,$$

where B_0 is a positive constant, and consider the following auxiliary linear parabolic equation for any given $\ell \in \mathcal{D}(Q)$:

$$\begin{cases} \eta_{\varepsilon,t} + b_\varepsilon \Delta \eta_\varepsilon + \mathbf{v} \cdot \nabla \eta_\varepsilon = \ell \quad \text{in } Q, \\ \dfrac{\partial \eta_\varepsilon}{\partial \nu} + n_0 \eta_\varepsilon = 0 \quad \text{on } \Sigma, \\ \eta_\varepsilon(T, \cdot) = 0 \quad \text{on } \Omega. \end{cases} \tag{5.4}$$

By the general theory (cf. [11;Chapter 3]) of linear parabolic equations this problem has a unique solution $\eta_\varepsilon \in H^{2+\alpha,1+\alpha/2}(\bar{Q})$ for a certain number $0 < \alpha < 1$.

Now we are going to show that $\{\eta_\varepsilon\}$ is bounded in $L^\infty(0,T;V)$ and $\{b_\varepsilon^{1/2}\Delta\eta_\varepsilon\}$ is bounded in $L^2(Q)$. To this end we prove:

Lemma 5.1

$$\begin{aligned} &|\eta_\varepsilon(s)|_V^2 + 2\int_s^T \int_\Omega b_\varepsilon |\Delta \eta_\varepsilon|^2 dx dt \\ \leq\ & \{6|\mathbf{v}|_{C^1(\overline{Q})^3} + 1\} \int_s^T |\eta_\varepsilon(t)|_V^2 dx dt + n_0 \int_s^T \int_\Gamma \mathbf{v} \cdot \nabla(\eta_\varepsilon^2) d\Gamma dt + \int_0^T |\ell(t)|_V^2 dt \end{aligned} \tag{5.5}$$

for all $s \in [0,T]$ and $\varepsilon \in (0,1]$.

Proof. Multiplying the first equation in (5.4) by $\Delta\eta_\varepsilon$ and integrating over Ω with respect to x, we get for any fixed time t

$$\int_\Omega \frac{\partial \eta_\varepsilon}{\partial t} \Delta \eta_\varepsilon dx + \int_\Omega |b_\varepsilon||\Delta\eta_\varepsilon|^2 dx + \int_\Omega (\mathbf{v}\cdot\nabla\eta_\varepsilon)\Delta\eta_\varepsilon dx = -\int_\Omega \nabla \ell \cdot \nabla \eta_\varepsilon dx. \tag{5.6}$$

Here

$$\begin{aligned} \int_\Omega \frac{\partial \eta_\varepsilon}{\partial t}\Delta\eta_\varepsilon dx &= -\int_\Omega \left(\nabla \frac{\partial \eta_\varepsilon}{\partial t} \cdot \nabla \eta_\varepsilon\right) dx + \int_\Gamma \frac{\partial \eta_\varepsilon}{\partial t}\frac{\partial \eta_\varepsilon}{\partial \nu} d\Gamma \\ &= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\eta_\varepsilon|^2 dx - \frac{n_0}{2}\frac{d}{dt}\int_\Gamma |\eta_\varepsilon|^2 d\Gamma \\ &= -\frac{1}{2}\frac{d}{dt}|\eta_\varepsilon(t)|_V^2. \end{aligned}$$

and

$$\begin{aligned} &-\int_\Omega (\mathbf{v}\cdot\nabla\eta_\varepsilon)\Delta\eta_\varepsilon dx \\ =\ & \int_\Omega \nabla(\mathbf{v}\cdot\nabla\eta_\varepsilon)\cdot\nabla\eta_\varepsilon dx - \int_\Gamma (\mathbf{v}\cdot\nabla\eta_\varepsilon)\frac{\partial\eta_\varepsilon}{\partial\nu} d\Gamma \\ =\ & \int_\Omega \sum_{i,j=1}^3 \left\{ \frac{\partial v_j}{\partial x_i}\frac{\partial \eta_\varepsilon}{\partial x_j}\frac{\partial \eta_\varepsilon}{\partial x_i} + v_j \frac{\partial^2 \eta_\varepsilon}{\partial x_j \partial x_i}\frac{\partial \eta_\varepsilon}{\partial x_i}\right\} dx + n_0 \int_\Gamma (\mathbf{v}\cdot\nabla\eta_\varepsilon)\eta_\varepsilon d\Gamma \\ \leq\ & 3|\mathbf{v}|_{C^1(\overline{Q})^3}\int_\Omega |\nabla\eta_\varepsilon|^2 dx + \int_\Omega \mathbf{v}\cdot\nabla\left(\frac{1}{2}|\nabla\eta_\varepsilon|^2\right) dx + \frac{n_0}{2}\int_\Gamma \mathbf{v}\cdot\nabla(\eta_\varepsilon^2) d\Gamma. \end{aligned}$$

Now, by using conditions (1.1) and (1.3) we see that

$$\begin{aligned} \int_\Omega \mathbf{v}\cdot\nabla\left(\frac{1}{2}|\nabla\eta_\varepsilon|^2\right) dx &= \int_\Omega \operatorname{div}\left(\frac{1}{2}|\nabla\eta_\varepsilon|^2\mathbf{v}\right) dx \\ &= \int_\Gamma \frac{1}{2}|\nabla\eta_\varepsilon|^2(\mathbf{v}\cdot\nu) d\Gamma \\ &= 0. \end{aligned}$$

It follows from (5.6) with the above estimates that

$$-\frac{1}{2}\frac{d}{dt}|\eta_\varepsilon|_V^2 + \int_\Omega |b_\varepsilon||\Delta\eta_\varepsilon|^2 dx \le (3|\mathbf{v}|_{C^1(\overline{Q})^3} + \frac{1}{2})|\eta_\varepsilon|_V^2 + \frac{1}{2}|\ell|_V^2 + n_0 \int_\Gamma \mathbf{v}\cdot\nabla(\eta_\varepsilon^2)d\Gamma.$$

for all $t \in [0,T]$, so that (5.5) is obtained by integrating this inequality over $[s,T]$ in time. □

Lemma 5.2 *There is a positive constant B_1 such that*

$$\int_\Gamma \mathbf{v}(t)\cdot\nabla(\eta_\varepsilon^2(t))d\Gamma \le B_1|\eta_\varepsilon(t)|_{L^2(\Gamma)}^2 \quad \textit{for all } t\in[0,T] \textit{ and } \varepsilon\in(0,1].$$

Proof. Our geometric condition ($\star$) ensures that there is a finite open covering $\{U_k\}_{k=1}^N$ of Γ such that a local coordinate transformation $y := (y_1,y_2,y_3) = X_k(x) := (X_{k1}(x), X_{k2}(x), X_{k3}(x))$ from U_k onto an open subset $\tilde{U}_k$ of the y-space satisfying properties

- $X_k(U_k\cap\Omega) \subset Y := \{y; y_3 > 0\}$ and $X_k(U_k\cap\Gamma) \subset \partial Y := \{y; y_3 = 0\}$ for all $k = 1,2,...,N$.
- $\dfrac{\partial}{\partial\nu} = A_k(y')\dfrac{\partial}{\partial y_3}$ on $U_k\cap\Gamma$, where $y' = (y_1,y_2,0)$ and $A_k(y')$ is a positive bounded function of C^1 on $U_k\cap\Gamma$ for all $k = 1,2,...N$.

Also we take a partition of unity $\{\phi_k\}$ on Γ, namely

$$\phi_k \in C_0^\infty(\mathbf{R}^3),\ \ \mathrm{supp}(\phi_k)\subset U_k,\ 1\le k\le N,\quad \sum_{k=1}^N \phi_k = 1 \ \text{ on } \Gamma,$$

and use the relation $d\Gamma = J_k(y')dy'$ on $U_k\cap\Gamma$ for $k = 1,2,...,N$, where J_k is a positive and bounded function of C^1 on $U_k\cap\Gamma$.

We have for any fixed $t\in[0,T]$ that

$$\begin{aligned}\int_\Gamma \mathbf{v}\cdot\nabla(\eta_\varepsilon^2)d\Gamma &= \sum_{i=1}^3\int_\Gamma v_i\frac{\partial\eta_\varepsilon^2}{\partial x_i}d\Gamma = \sum_{i=1}^3\left\{\sum_{k=1}^N\int_\Gamma \phi_k v_i\frac{\partial\eta_\varepsilon^2}{\partial x_i}d\Gamma\right\}\\ &= \sum_{i,j=1}^3\left\{\sum_{k=1}^N\int_{\mathbf{R}^2}\phi_k v_i\frac{\partial X_{kj}}{\partial x_i}\frac{\partial\eta_\varepsilon^2}{\partial y_j}J_j dy'\right\}.\end{aligned}$$

Therefore, for $w_{ijk} := \phi_k v_i(\partial X_{kj}/\partial x_i)J_j$, it is enough to estimate the integral

$$I_{ik} := \int_{\mathbf{R}^2}\sum_{j=1}^3 w_{ijk}\frac{\partial\eta_\varepsilon^2}{\partial y_j}dy'.$$

In fact, we see that

$$I_{ik} = -\int_{\mathbf{R}^2}\left\{\frac{\partial w_{i1k}}{\partial y_1} + \frac{\partial w_{i2k}}{\partial y_2}\right\}\eta_\varepsilon^2 dy' + \int_{\mathbf{R}^2} w_{i3k}\frac{\partial\eta_\varepsilon^2}{\partial y_3}dy'.$$

Here, noting that

$$\frac{\partial \eta_\varepsilon^2}{\partial y_3} = \frac{1}{A_j}\frac{\partial \eta_\varepsilon^2}{\partial \nu} = -\frac{2n_0}{A_j}|\eta_\varepsilon|^2,$$

we get from the above expression of I_{ik} that

$$|I_{ik}| \le B_2|\eta_\varepsilon|^2_{L^2(\Gamma)}$$

for all $i,k = 1,2,3$, and B_2 is a positive constant independent of i,j,k. Thus the required estimate holds with $B_1 = 6B_2$. □

Now, from Lemmas 5.1 and 5.2 it follows that

$$|\eta_\varepsilon(s)|_V^2 + \int_s^T \int_\Omega b_\varepsilon |\Delta\eta_\varepsilon|^2 dxdt \le B_3 \int_s^T \{|\eta_\varepsilon|_V^2 + |\ell|_V^2\} dt \tag{5.7}$$

for all $s \in [0,T]$ and all $\varepsilon \in (0,1]$. Applying the Gronwall's inequality to (5.7), we have the uniform estimate

$$\sup_{0\le t\le T} |\eta_\varepsilon(t)|_V^2 + \int_Q b_\varepsilon |\Delta\eta_\varepsilon|^2 dxdt \le B_4, \tag{5.8}$$

where B_4 is a positive constant independent of $\varepsilon \in (0,1]$.

By (5.8) we can find a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ and a function $\eta \in V$ such that

$$\eta_{\varepsilon_n} \to \eta \quad \text{weakly in } L^2(Q),$$
$$\nabla\eta_{\varepsilon_n} \to \nabla\eta \quad \text{weakly in } L^2(Q)^3 \text{ as } n \to 0.$$

Taking η_{ε_n} as a test function η in (5.2), we have

$$\begin{aligned}
&-\int_Q (u_1-u_2)\{\eta_{\varepsilon_n,t} + b\Delta\eta_{\varepsilon_n} + \mathbf{v}\cdot\nabla\eta_{\varepsilon_n}\} dxdt \\
&= -\int_Q (u_1-u_2)\{\eta_{\varepsilon_n,t} + b_{\varepsilon_n}\Delta\eta_{\varepsilon_n} + \mathbf{v}\cdot\nabla\eta_{\varepsilon_n}\} dxdt + \int_Q (u_1-u_2)(b_{\varepsilon_n}-b)\Delta\eta_{\varepsilon_n} dxdt \\
&= -\int_Q (u_1-u_2)\ell dxdt + \int_Q (u_1-u_2)(b_{\varepsilon_n}-b)\Delta\eta_{\varepsilon_n} dxdt.
\end{aligned}$$

Moreover, passing to the limit in n yields that

$$\begin{aligned}
&\left|\int_Q (u_1-u_2)(b_{\varepsilon_n}-b)\Delta\eta_{\varepsilon_n} dxdt\right| \\
&\le \left\{\int_Q |u_1-u_2|^2 |b_{\varepsilon_n}-b| dxdt\right\}^{\frac{1}{2}} \left\{\int_Q |b_{\varepsilon_n}-b||\Delta\eta_{\varepsilon_n}|^2 dxdt\right\}^{\frac{1}{2}} \\
&\le \left\{\int_Q |u_1-u_2|^2 |b_{\varepsilon_n}-b| dxdt\right\}^{\frac{1}{2}} B_4 \\
&\to 0.
\end{aligned}$$

Therefore

$$\int_Q (u_1-u_2)\ell dxdt = 0 \quad \text{for all } \ell \in \mathcal{D}(Q),$$

which implies that $u_1 = u_2$ a.e. on Q. □

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